

COMPACTNESS ISSUES AND BUBBLING PHENOMENA FOR THE PRESCRIBED GAUSSIAN CURVATURE EQUATION ON THE TORUS

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ABSTRACT. In the spirit of the previous paper [5], where we dealt with the case of a closed Riemann surface (M, g_0) of genus greater than one, here we study the behaviour of the conformal metrics g_λ of prescribed Gauss curvature $K_{g_\lambda} = f_0 + \lambda$ on the torus, when the parameter λ tends to one of the boundary points of the interval of existence of g_λ , and we characterize their “bubbling behavior” as in [5].

1. INTRODUCTION

Consider a closed, connected Riemann surface M , whose Euler characteristic $\chi(M)$ is zero, endowed with a smooth background metric g_0 . In view of the uniformization theorem, it is possible to assume that the Gauss curvature K_{g_0} of g_0 vanishes identically.

The prescribed Gauss curvature equation, which links the curvature of g_0 to the curvature K_g of a conformal metric $g = e^{2u}g_0$, then reads as

$$K_g = -e^{-2u} \Delta_{g_0} u$$

Moreover, for convenience, we normalize the volume of (M, g_0) to unity.

Consider a smooth non-constant function $f_0 : M \rightarrow \mathbb{R}$ with $\max_{p \in M} f_0(p) = 0$, all of whose maximum points are non-degenerate, and define for $\lambda \in \mathbb{R}$

$$f_\lambda := f_0 + \lambda$$

A natural question is to understand for which values λ the function f_λ is the Gauss curvature of a metric conformal to g_0 . That is equivalent to ask for which values of λ , the equation

$$(1.1) \quad -\Delta_{g_0} u = f_\lambda e^{2u}$$

admits a solution. The paper [13] completely answers the question, giving necessary and sufficient conditions for solving the equation above. More precisely, equation (1.1) has a solution if and only if

$$\int_M f_\lambda d\mu_{g_0} = \int_M f_0 d\mu_{g_0} + \lambda < 0$$

and f_λ is sign changing. (Recall that the volume is equal to one.) Thus, by taking account of the assumptions made on f_0 , we find that equation (1.1) is solvable if and only if

$$0 = -\max_M f_0 < \lambda < -\overline{f_0},$$

Date: September 18, 2014.

Supported by SNF grant 200021_140467 / 1.

where $\overline{f_0} := \int_M f_0 d\mu_{g_0}$. Set

$$\Lambda := (0, -\overline{f_0}), \quad -\overline{f_0} := \lambda_{max}.$$

Our goal in this paper is to study the behaviour of the set of solutions of (1.1) when λ approaches either 0 or λ_{max} , a problem left open in [13] and which we solve by means of a blow-up analysis in the spirit of [5]. Our main results are the following:

Theorem 1.1. *Let $f_0 \leq 0$ be a smooth, non-constant function, all of whose maximum points p_0 are non-degenerate with $f_0(p_0) = 0$, and for $\lambda \in \mathbb{R}$ let $f_\lambda = f_0 + \lambda$.*

Then there exists a sequence $\lambda_n \downarrow 0$, a sequence u_n of solutions of the equation

$$-\Delta_{g_0} u_n = f_{\lambda_n} e^{2u_n}$$

and there exists $I \in \mathbb{N}$ such that, for suitable $p_n^{(i)} \rightarrow p_\infty^{(i)} \in M$ with $f_0(p_\infty^{(i)}) = 0$, $1 \leq i \leq I$, we obtain $u_n(p_n^{(i)}) \rightarrow +\infty$ and one of the following:

i) $u_n \rightarrow -\infty$ locally uniformly on compact domains of $M_\infty := M \setminus \{p_\infty^{(i)}; 1 \leq i \leq I\}$

ii) For suitable $r_n^{(i)} \downarrow 0$, the following holds:

a) We have smooth convergence $u_n \rightarrow u_\infty$ locally on M_∞ and u_∞ induces a complete metric $g_\infty = e^{2u_\infty} g_0$ on M_∞ of finite total curvature $K_{g_\infty} = f_0$.

b) For each $1 \leq i \leq I$, either 1) there holds $r_n^{(i)} / \sqrt{\lambda_n} \rightarrow 0$ and in local conformal coordinates around $p_n^{(i)}$ we have

$$w_n(x) := u_n(r_n^{(i)} x) - u_n(0) + \log 2 \rightarrow w_\infty(x) = \log \left(\frac{2}{1 + |x|^2} \right)$$

smoothly locally in \mathbb{R}^2 , where w_∞ induces a spherical metric $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ of curvature $K_{g_\infty} = 1$ on \mathbb{R}^2 , or 2) we have $r_n^{(i)} = \sqrt{\lambda_n}$, and in local conformal coordinates around $p_\infty^{(i)}$ with a constant $c_\infty^{(i)}$ there holds

$$w_n(x) = u_n(r_n^{(i)} x) + \log(\lambda_n) + c_\infty^{(i)} \rightarrow w_\infty(x)$$

smoothly locally in \mathbb{R}^2 , where the metric $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ on \mathbb{R}^2 has finite volume and finite total curvature with $K_{g_\infty}(x) = 1 + (Ax, x)$, where $A = \frac{1}{2} \text{Hess} f_0(p_\infty^{(i)})$.

Moreover, we have

Theorem 1.2. *Let $f_0 \leq 0$ be a smooth, non-constant function. For $\lambda \in \mathbb{R}$ set*

$$(1.2) \quad \mathcal{C}_\lambda := \left\{ u \in H^1(M; g_0) : \int_M u d\mu_{g_0} = 0 = \int_M f_\lambda e^{2u} d\mu_{g_0} \right\}.$$

Then for any arbitrary sequence $(\lambda_n)_n \subset \Lambda$ such that $\lambda_n \uparrow \lambda_{max}$ for $n \rightarrow +\infty$, we have that:

i) there exists a sequence of minimizers $w_n \in \mathcal{C}_{\lambda_n}$ of the Dirichlet energy such that:

$$w_n \rightarrow 0 \quad \text{in } C^{2,\alpha}(M)$$

for any $\alpha \in [0, 1)$.

ii) there exists a sequence of solutions u_n to equation

$$-\Delta_{g_0} u_n = f_{\lambda_n} e^{2u_n}$$

such that $u_n \rightarrow -\infty$ uniformly on the whole M .

Observation 1.3. *We remark that in Theorem 1.2 no assumptions have been made on the nature of the points of maximum of the function f_0 .*

Observation 1.4. *In contrast to [5], in the present paper the monotonicity of the energy of the solutions u_λ as a function of λ is not obvious. The proof of this fact is perhaps the main new technical achievement in the present work.*

ACKNOWLEDGMENTS

I would like to thank Michael Struwe for the guidance through the project which led to this paper.

2. SOME NOTATION AND PRELIMINARY RESULTS

In the following section we will recall some well-known results about the existence of solutions to equation (1.1) and introduce some notation and concepts used through the rest of the paper. For further details we refer to [13].

For $\lambda \in \mathbb{R}$ consider the set \mathcal{C}_λ defined by (1.2). Note that for $\lambda \in (0, -\min_M f_0)$ the function f_λ is sign changing and hence $\mathcal{C}_\lambda \neq \emptyset$. On the other hand, $\mathcal{C}_\lambda = \emptyset$ for $\lambda \leq 0$ or $\lambda \geq -\min_M f_0$.

The constraints defining \mathcal{C}_λ are natural; the first allows to apply the direct methods, the second one is motivated by the Gauss-Bonnet Theorem.

Lemma 2.1. *For $\lambda \in (0, -\min_M f_0)$ the set \mathcal{C}_λ is a C^∞ -Banach manifold.*

Proof. Define $G^\lambda : H^1(M; g_0) \rightarrow \mathbb{R}^2$ by letting

$$G^\lambda(u) := \left(\int_M u d\mu_{g_0}, \int_M f_\lambda e^{2u} d\mu_{g_0} \right).$$

Then G^λ is smooth and its first derivative is

$$DG^\lambda(u)[v] = \left(\int_M v d\mu_{g_0}, 2 \int_M f_\lambda v e^{2u} d\mu_{g_0} \right).$$

Notice that $(G^\lambda)^{-1}(0) = \mathcal{C}_\lambda$. Pick $u \in \mathcal{C}_\lambda$. If we compute $DG^\lambda(u)[v]$ with $v \equiv 1$ and then with $v = f_\lambda$, we get two vectors of \mathbb{R}^2 which are linearly independent; therefore $DG^\lambda(u)$ is surjective. Since we are in the Hilbert space $H^1(M; g_0)$, we have that it is splitted by the kernel of $DG^\lambda(u)$. It follows that G^λ is a submersion at $u \in \mathcal{C}_\lambda$ and then that \mathcal{C}_λ is a smooth manifold. (For further details we refer to [22]). The lemma is proved. \square

In order to find solutions to equation (1.1) for $\lambda \in \Lambda$, we minimize the Dirichlet energy E

$$H^1(M; g_0) \ni u \xrightarrow{E} \int_M |\nabla u|_{g_0}^2 d\mu_{g_0}$$

in \mathcal{C}_λ . The energy E is coercive on \mathcal{C}_λ in view of Poincaré's inequality and sequentially weakly lower semicontinuous. Furthermore, \mathcal{C}_λ is weakly sequentially closed as can easily be shown by means of Moser-Trudinger's inequality. Hence the direct method of the calculus of variation applies and for each $\lambda \in \Lambda$ there exists a minimizer $w_\lambda \in \mathcal{C}_\lambda$.

But in the course of the proof of Lemma 2.1 we have seen that G^λ is a submersion at any point of \mathcal{C}_λ : therefore we can apply the Lagrange multipliers rule and obtain

$$(2.1) \quad 2 \int_M (\nabla w_\lambda, \nabla v)_{g_0} d\mu_{g_0} = \sigma \int_M v d\mu_{g_0} + 2\mu \int_M f_\lambda v e^{2w_\lambda} d\mu_{g_0}$$

for every $v \in H^1(M; g_0)$ with suitable $\sigma, \mu \in \mathbb{R}$. Choosing $v \equiv 1$, we obtain

$$0 = \sigma \int_M d\mu_{g_0} + 2\mu \int_M f_\lambda e^{2w_\lambda} d\mu_{g_0};$$

hence $\sigma = 0$, because $w_\lambda \in \mathcal{C}_\lambda$. Notice that, by regularity arguments (see [13] for the details), $w_\lambda \in C^\infty(M)$ and hence $v \equiv e^{-2w_\lambda} \in H^1(M; g_0)$. For this choice of testing function (2.1) gives

$$0 \geq -2 \int_M |\nabla w_\lambda|_{g_0}^2 e^{-2w_\lambda} d\mu_{g_0} = \mu \int_M f_\lambda d\mu_{g_0}.$$

If

$$\int_M |\nabla w_\lambda|_{g_0}^2 e^{-2w_\lambda} d\mu_{g_0} = 0,$$

we get $w_\lambda \equiv \text{constant}$, which is a contradiction, since in \mathcal{C}_λ there are no constant functions for $\lambda \in \Lambda$. Therefore, since $\int_M f_\lambda d\mu_{g_0} < 0$ for $\lambda \in \Lambda$, we obtain

$$\mu = \mu(\lambda) = -2 \frac{\int_M |\nabla w_\lambda|_{g_0}^2 e^{-2w_\lambda} d\mu_{g_0}}{\int_M f_\lambda d\mu_{g_0}} > 0.$$

As a consequence,

$$(2.2) \quad u_\lambda := w_\lambda + 1/2 \log \mu(\lambda)$$

classically solves (1.1).

For the continuation of our analysis and for technical reasons which will become evident later, it is convenient to introduce for $\lambda \in \mathbb{R}$ the set

$$\mathcal{E}_\lambda := \left\{ u \in H^1(M; g_0) : 0 = \int_M f_\lambda e^{2u} d\mu_{g_0} \right\},$$

defined by a single constraint only.

As above, it can be seen that $\mathcal{E}_\lambda \neq \emptyset$ if and only if $\lambda \in (0, -\min_M f_0)$ and that it is a C^∞ -Banach manifold.

A priori it is not clear if we may expect that the Dirichlet energy E attains a minimum in \mathcal{E}_λ ; however an elementary argument shows that for $\lambda \in (0, \lambda_{max})$ we have

$$E(u_\lambda) = \min_{v \in \mathcal{E}_\lambda} E(v) = \min_{v \in \mathcal{C}_\lambda} E(v)$$

where u_λ is defined by (2.2). Indeed, for any $v \in \mathcal{E}_\lambda$, we have $v - \bar{v} \in \mathcal{C}_\lambda$ and $E(v) = E(v - \bar{v})$, where $\bar{v} := \int_M v d\mu_{g_0}$.

Notice finally that for $\lambda = \lambda_{max}$, $u \equiv \text{constant}$ belongs to \mathcal{E}_λ and it minimizes the energy (which is zero). Furthermore, for $\lambda \in (\lambda_{max}, -\min_M f_0)$, it is always true that the energy E , even though it does not admit a minimum, is non negative. That suggests to define the following function:

$$(2.3) \quad \beta_\lambda := \begin{cases} \int_M |\nabla u_\lambda|_{g_0}^2 d\mu_{g_0} = E(u_\lambda) & \text{if } \lambda \in (0, \lambda_{max}) \\ 0 & \text{if } \lambda \in [\lambda_{max}, -\min_M f_0]. \end{cases}$$

In the next sections, we study the properties of β_λ and use this information to prove, respectively, Theorem 1.1 and Theorem 1.2.

3. PROOF OF THEOREM 1.1

In this section we will analyse the behaviour of the set of solutions to equation (1.1) when the parameter λ approaches zero and we will prove Theorem 1.1.

The first result is quite elementary but it shows that in an arbitrary neighborhood of zero the function β_λ can achieve arbitrarily large values. More precisely, we can state:

Lemma 3.1. $\limsup_{\lambda \downarrow 0, \lambda \in \Lambda} \beta_\lambda = +\infty$.

Proof. Assume by contradiction that there exists $\delta \in \Lambda$ such that $\sup_{\lambda \in (0, \delta)} \beta_\lambda < +\infty$. Choose a sequence $(\lambda_n)_n \subset (0, \delta)$ which converges to zero as $n \rightarrow +\infty$. Thus we have $\int_M |\nabla w_{\lambda_n}|_{g_0}^2 d\mu_{g_0} < +\infty$ uniformly in n , where $w_{\lambda_n} \in \mathcal{C}_{\lambda_n}$ is a minimizer of the energy E . Therefore, since the average of w_{λ_n} is zero, we have, up to subsequences, that $w_{\lambda_n} \rightharpoonup w_0$ weakly in $H^1(M; g_0)$ and $e^{2w_{\lambda_n}} \rightarrow e^{2w_0}$ strongly in L^1 . Thus

$$0 = \int_M f_{\lambda_n} e^{2w_{\lambda_n}} d\mu_{g_0} \rightarrow \int_M f_0 e^{2w_0} d\mu_{g_0}$$

and $w_0 \in \mathcal{E}_0 = \emptyset$. The contradiction proves the Lemma. \square

In the following, we are going to construct a suitable comparison function belonging to the manifold \mathcal{E}_λ , which will give a control on the rate of blow-up of the minimum of the energy. This is the content of the next proposition, but before we need:

Lemma 3.2. *There exists $L > 0$ such that for any $\lambda < -\min_M f_0$ and for any $p \in M$ point of maximum of f_0 we have*

- i. $\frac{\sqrt{\lambda}}{L} < 1$
- ii. $f_0(x) > -\frac{\lambda}{2}$ on $B_{\frac{\sqrt{\lambda}}{L}}(0) \subset \mathbb{R}^2$,

where x are suitable local conformal coordinates around $p \simeq 0$.

Proof. Fix a point of maximum p_i of f_0 . Then, by choosing local conformal coordinates x around $p_i \simeq 0$, we have

$$f_0(x) = \frac{1}{2} D^2 f_0(0)[x, x] + O(|x|^3) \text{ in } B_1(0) \subset \mathbb{R}^2$$

From the beginning we may assume that $\frac{1}{2} D^2 f_0(0)[x, x] \geq -c_1 |x|^2$, where $c_1 > 0$. Then, for $x \in B_1(0)$, we have

$$f_0(x) \geq -c_1 |x|^2 - c_2 |x|^3 \geq -c(|x|^2 + |x|^3),$$

with $c_2 > 0$ and $c := \max(c_1, c_2) > 0$.

Pick $\lambda > 0$ and $L_i > 0$ to be determined later, such that $\sqrt{\lambda}/L_i < 1$, namely $\lambda < L_i^2$. Then, on the ball $B_{\frac{\sqrt{\lambda}}{L_i}}(0)$ we get

$$f_0(x) > -c \left(\frac{\lambda}{L_i^2} + \frac{\lambda^{3/2}}{L_i^3} \right) \geq -\frac{\lambda}{2}$$

where the last inequality holds if we choose $L_i^2 \geq 4c$. Choose $L_i \gg 0$ so that $-\min_M f_0 < L_i^2$. Taking $L := \max_{p_i} L_i$, we obtain the desired result. \square

Proposition 3.3. *For any $0 < \sigma \leq 1$ there exists $\lambda_\sigma < 1$, $\lambda_\sigma \in \Lambda$, such that for any $0 < \lambda \leq \lambda_\sigma$ there holds:*

$$(3.1) \quad \beta_\lambda \leq 2\pi M_0 (\sigma + 2)^2 \log(1/\lambda)$$

where M_0 is a constant which depends only on (M, g_0) and the function f_0 .

Proof. Choose $p_0 \in M$ such that $f_0(p_0) = 0$ and choose conformal coordinates x as in the previous Lemma so that

$$f_0(x) + \lambda \geq \frac{\lambda}{2}, \quad x \in B_{\frac{\sqrt{\lambda}}{L}}(0)$$

for any $\lambda < -\min_M f_0$. Locally we can write $g_0 = e^{2v_0} g_{\mathbb{R}^2}$ where $v_0 \in C^\infty(\overline{B_1(0)})$ and $v_0(0) = 0$. Fix $\lambda \in \Lambda$ with $\lambda < 1$. Define the function $\varphi(\lambda) : M \rightarrow \mathbb{R}$ as

$$(3.2) \quad \varphi(\lambda)(x) = \begin{cases} \log\left(\frac{\sqrt{\lambda}}{L|x|}\right), & \frac{\lambda^{3/2}}{L} \leq |x| \leq \frac{\sqrt{\lambda}}{L} \\ \log\left(\frac{1}{\lambda}\right), & |x| \leq \frac{\lambda^{3/2}}{L} \\ 0, & \frac{\sqrt{\lambda}}{L} \leq |x| \leq 1 \end{cases}$$

extended to zero on the rest of M . We have $\varphi(\lambda) \in H^1(M; g_0)$ and f_λ is positive on the support of $\varphi(\lambda)$.

Consider the continuous function $z : \mathbb{R} \rightarrow \mathbb{R}$ defined by $z(\alpha) = \int_M f_\lambda e^{2\alpha\varphi(\lambda)} d\mu_{g_0}$; then $z(0) < 0$ and $\lim_{\alpha \rightarrow +\infty} z(\alpha) = +\infty$; thus there exists $\alpha = \alpha(\lambda) \in (0, +\infty)$ where

$$0 = z(\alpha) = \int_M f_\lambda e^{2\alpha\varphi(\lambda)} d\mu_{g_0},$$

that is, $\alpha\varphi(\lambda) \in \mathcal{E}_\lambda$.

We can give a more precise estimate of α , as follows. Recall that $\text{Vol}(M; g_0) = 1$, therefore

$$\begin{aligned} 0 = \int_M f_\lambda e^{2\alpha\varphi(\lambda)} d\mu_{g_0} &\geq \lambda/2 \int_{B_{\frac{\sqrt{\lambda}}{L}}(0)} e^{2\alpha\varphi(\lambda)} e^{2v_0} dx - \|f_0\|_\infty \\ &> \lambda/2 \int_{B_{\frac{\lambda^{3/2}}{L}}(0)} e^{2\alpha \log(1/\lambda)} e^{2v_0} dx - \|f_0\|_\infty. \end{aligned}$$

Let $m_0 := \min_{B_1(0)} e^{2v_0}$ and $M_0 := \max_{B_1(0)} e^{2v_0}$. We obtain:

$$\frac{m_0 \pi}{2} \frac{\lambda^{4-2\alpha}}{L^2} \leq \|f_0\|_\infty$$

or equivalently

$$0 < \alpha \leq \frac{\log\left(\frac{2L^2 \|f_0\|_\infty}{m_0 \pi}\right)}{2 \log(1/\lambda)} + 2.$$

Given $0 < \sigma \leq 1$, there exists $\lambda_\sigma < 1$, $\lambda_\sigma \in \Lambda$, such that for any $0 < \lambda \leq \lambda_\sigma$ we have $\frac{\log\left(\frac{2L^2 \|f_0\|_\infty}{m_0 \pi}\right)}{2 \log(1/\lambda)} < \sigma$. Hence

$$\alpha^2 \leq (\sigma + 2)^2.$$

Next we have:

$$\int_M |\nabla \varphi(\lambda)|_{g_0}^2 d\mu_{g_0} = \int_{B_{\frac{\sqrt{\lambda}}{L}}(0) \setminus B_{\frac{\lambda^{3/2}}{L}}(0)} |x|^{-2} e^{2v_0} dx;$$

hence

$$(3.3) \quad m_0 2\pi \log(1/\lambda) \leq \int_M |\nabla \varphi(\lambda)|_{g_0}^2 d\mu_{g_0} \leq M_0 2\pi \log(1/\lambda).$$

We conclude

$$\beta_\lambda \leq \alpha^2 \int_M |\nabla \varphi(\lambda)|_{g_0}^2 d\mu_{g_0} \leq 2\pi M_0 (\sigma + 2)^2 \log(1/\lambda),$$

which proves the Proposition. \square

From Proposition 3.3, by means of elliptic estimates we obtain uniform L^∞ -bounds for the set of solutions of (1.1), away from the boundary of Λ . More precisely:

Proposition 3.4. *Fix $0 < \sigma \leq 1$ and let λ_σ be as in Proposition (3.3). Then for any $\lambda^* \in (0, \lambda_\sigma)$ we have*

$$(3.4) \quad \sup_{\lambda^* \leq \lambda \leq \lambda_\sigma} \|u_\lambda\|_\infty < +\infty.$$

Observation 3.5. *Obviously, the estimate above can be improved by replacing the L^∞ norm with "higher" norms (use a bootstrap argument), but in the rest of the paper the estimate above will turn out to be sufficient for all our purposes.*

Proof of Proposition 3.4. Because of the Sobolev embedding, it is enough to prove that

$$\sup_{\lambda^* \leq \lambda \leq \lambda_\sigma} \|u_\lambda\|_{H^2} < +\infty$$

For $\lambda \in [\lambda^*, \lambda_\sigma]$ consider the minimizer $w_\lambda \in \mathcal{C}_\lambda$, which solves the equation

$$-\Delta_{g_0} w_\lambda = \mu(\lambda) f_\lambda e^{2w_\lambda}$$

where $\mu(\lambda) \in (0, \infty)$ is a Lagrange multiplier.

From (3.1), we have

$$\sup_{\lambda^* \leq \lambda \leq \lambda_\sigma} \beta_\lambda \leq C \log(1/\lambda^*).$$

Hence, using Poincaré's inequality, we obtain $\|w_\lambda\|_{H^1(M; g_0)} < C$ uniformly in λ . By the Moser-Trudinger's inequality, for every $p \geq 1$ then there holds:

$$\sup_{\lambda^* \leq \lambda \leq \lambda_\sigma} \int_M e^{pw_\lambda} d\mu_{g_0} < C(p) < \infty.$$

Our claim thus follows once we can give a lower and an upper bound for $\mu(\lambda)$. Inserting $v = f_\lambda$ in (2.1), we obtain

$$(3.5) \quad \int_M (\nabla w_\lambda, \nabla f_\lambda)_{g_0} d\mu_{g_0} = \mu(\lambda) \int_M (f_\lambda)^2 e^{2w_\lambda} d\mu_{g_0}.$$

Since $\lambda_\sigma < -\bar{f}_0$, we have $0 < c \leq \left(\int_M f_\lambda d\mu_{g_0} \right)^2$ uniformly in $\lambda \in [\lambda^*, \lambda_\sigma]$. Thus, by Hölder

$$c < \int_M (f_\lambda)^2 e^{2w_\lambda} d\mu_{g_0} \int_M e^{-2w_\lambda} d\mu_{g_0}.$$

Applying Moser-Trudinger's inequality, we get

$$c < C \int_M (f_\lambda)^2 e^{2w_\lambda} d\mu_{g_0} \exp \left(\frac{1}{4\pi} \int_M |\nabla w_\lambda|_{g_0}^2 d\mu_{g_0} \right).$$

Thus, we see that $\int_M (f_\lambda)^2 e^{2w_\lambda} d\mu_{g_0}$ for $\lambda \in [\lambda^*, \lambda_\sigma]$ is uniformly bounded away from zero and, from (3.5), we obtain

$$\mu(\lambda) \leq C(\|w_\lambda\|_{H^1(M; g_0)}) < C < \infty$$

uniformly in λ .

To see that $\mu(\lambda)$ is also away from zero, we argue by contradiction. Assume that $\inf_{\lambda^* \leq \lambda \leq \lambda_\sigma} \mu(\lambda) = 0$. Take a sequence $\lambda_n \in [\lambda^*, \lambda_\sigma]$ such that:

$$\mu(\lambda_n) \rightarrow 0$$

and $\lambda_n \rightarrow \lambda \in [\lambda^*, \lambda_\sigma]$. From the estimates above, we can assume, up to subsequences, that $w_{\lambda_n} \rightharpoonup w$ weakly in $H^1(M; g_0)$ and $e^{2w_{\lambda_n}} \rightarrow e^{2w}$ strongly in L^1 as $n \rightarrow \infty$. Recall that we have

$$\int_M (\nabla w_{\lambda_n}, \nabla v)_{g_0} d\mu_{g_0} = \mu(\lambda_n) \int_M f_{\lambda_n} v e^{2w_{\lambda_n}} d\mu_{g_0}$$

for any $v \in H^1(M; g_0)$. Passing to the limit $n \rightarrow \infty$ in this equation, we obtain

$$\int_M (\nabla w, \nabla v)_{g_0} d\mu_{g_0} = 0$$

for each $v \in H^1(M; g_0)$, that is w is harmonic. But then $w \equiv 0$ which is clearly impossible. Therefore, we have shown that for $\lambda \in [\lambda^*, \lambda_\sigma]$, $\mu(\lambda)$ is uniformly away from 0 and infinity.

In conclusion, we get a uniform bound in λ for

$$\|\Delta_{g_0} w_\lambda\|_{L^2} = \|\mu(\lambda) f_\lambda e^{2w_\lambda}\|_{L^2}.$$

Hence, by L^p -elliptic estimates (see for instance [13], p. 24), we have

$$\|w_\lambda\|_{H^2} \leq C \{\|w_\lambda\|_{H^1} + \|\Delta_{g_0} w_\lambda\|_{L^2}\} < C,$$

uniformly for $\lambda \in [\lambda^*, \lambda_\sigma]$. Recalling equation (2.2) and the bounds on $\mu(\lambda)$, the bound (3.4) follows. \square

Remark 3.6. The Proposition above is false when λ approaches zero. Indeed, an estimate like $\sup_{0 < \lambda \leq \delta} \max_M u_\lambda < \infty$ for some δ would lead, in view of Schauder's estimates, to a uniform $C^{2,\alpha}$ bound for u_λ , which clearly contradicts Lemma 3.1.

In the following we show that the function β_λ is monotone decreasing in a suitable right neighborhood of zero, which is crucial for our argument. As a consequence, β_λ will be differentiable almost everywhere.

Proposition 3.7. *There exists $\lambda_0 \leq \min\{1/2, -\overline{f_0}/2\}$ such that for any $\lambda^* \in (0, \lambda_0)$ there exists $\ell(\lambda^*) \in (\lambda^*, -\min_M f_0)$ such that for any $\lambda \in (\lambda^*, \ell(\lambda^*))$ we have*

$$\beta_\lambda < \beta_{\lambda^*}$$

Furthermore, choosing $\lambda \in (0, \lambda_0)$, $\lambda > \lambda^$, and defining $\ell(\lambda)$ as above, we have $\ell(\lambda) - \lambda \geq \tau = \tau(\lambda^*) > 0$ where τ is a constant not depending on λ .*

Corollary 3.8. *There exists $\lambda_0 \leq \min\{1/2, -\overline{f_0}/2\}$ such that β_λ is strictly monotone decreasing on the interval $(0, \lambda_0)$.*

In order to prepare for the proof of Proposition 3.7, define the map $I : H^1(M; g_0) \rightarrow \mathbb{R}$ by letting

$$(3.6) \quad I(u) := -\frac{\int_M f_0 e^{2u} d\mu_{g_0}}{\int_M e^{2u} d\mu_{g_0}}.$$

Note that for any $u \in H^1(M; g_0)$ there holds

$$(3.7) \quad u \in \mathcal{E}_{I(u)}.$$

Moreover, we have $I(u) \in (0, -\min_M f_0)$ and I is smooth with first derivative given by the following expression:

$$(3.8) \quad DI(u)[v] = -2 \frac{\int_M f_{I(u)} v e^{2u} d\mu_{g_0}}{\int_M e^{2u} d\mu_{g_0}} \quad u, v \in H^1(M; g_0).$$

Fix $0 < \lambda_0 \leq \min \{1/2, -\overline{f_0}/2\}$ and for $\lambda^* \in (0, \lambda_0)$ let

$$(3.9) \quad A(\lambda^*) := \sup_{\lambda^* \leq \lambda < \lambda_0} \sup_M e^{2u_\lambda}$$

and

$$(3.10) \quad a(\lambda^*) := \inf_{\lambda^* \leq \lambda < \lambda_0} \inf_M e^{2u_\lambda}.$$

Observe that in view of Proposition 3.4, the above functions are well defined if λ_0 is taken small enough, and that $0 < a(\lambda^*) \leq A(\lambda^*) < +\infty$. Finally, A is monotone decreasing and a is monotone increasing in λ^* .

We are ready to prove our Proposition.

3.1. Proof of Proposition 3.7.

Proof. Fix for convenience $\sigma = 1$ and let λ_σ as given in Proposition 3.3. Consider $\min \{\lambda_0, \lambda_\sigma\}$, which with a little abuse of notation we will still call λ_0 .

We consider $\lambda^* \in (0, \lambda_0)$ and $\beta_{\lambda^*} = \int_M |\nabla u^*|_{g_0}^2 d\mu_{g_0}$, where we have used the abbreviation $u^* \equiv u_{\lambda^*}$. We also set $\varphi^* \equiv \varphi(\lambda^*)$, where $\varphi(\lambda^*)$ is the comparison function defined by the equation (3.2). (We recall that $\lambda^* < \lambda_0 \leq 1/2 < 1$, therefore φ^* is well defined.) Thus, we have inequality (3.3) and

$$\begin{aligned} \int_M (\nabla u^*, \nabla \varphi^*)_{g_0} d\mu_{g_0} &= \int_M f_{\lambda^*} \varphi^* e^{2u^*} d\mu_{g_0} \\ &> \frac{\lambda^*}{2} \log(1/\lambda^*) \int_{B_{\frac{(\lambda^*)^{3/2}}{L}}(0)} e^{2u^*} e^{2v_0} dx, \end{aligned}$$

since $f_{\lambda^*} \varphi^* \geq 0$ and since $f_{\lambda^*} \geq \lambda^*/2$ in the ball $B_{\frac{\sqrt{\lambda^*}}{L}}(0)$. Observing that

$$(3.11) \quad \int_{B_{\frac{(\lambda^*)^{3/2}}{L}}(0)} e^{2u^*} e^{2v_0} dx \geq \frac{m_0 \pi}{L^2} (\lambda^*)^3 a(\lambda^*)$$

where $a(\lambda^*)$ is defined by (3.10) and $m_0 = \min_{B_1(0)} e^{2v_0}$ as above, we obtain

$$(3.12) \quad \begin{aligned} \int_M (\nabla u^*, \nabla \varphi^*)_{g_0} d\mu_{g_0} &= \int_M f_{\lambda^*} \varphi^* e^{2u^*} d\mu_{g_0} \\ &> \frac{m_0 \pi}{2L^2} \log(1/\lambda^*) (\lambda^*)^4 a(\lambda^*) > 0. \end{aligned}$$

Moreover, using equations (3.1) and (3.3), from Hölder's inequality we deduce

$$\int_M (\nabla u^*, \nabla \varphi^*)_{g_0} d\mu_{g_0} \leq 6\pi M_0 \log(1/\lambda^*).$$

Hence, defining

$$(3.13) \quad \varepsilon^* := 2 \frac{\int_M (\nabla u^*, \nabla \varphi^*)_{g_0} d\mu_{g_0}}{\int_M |\nabla \varphi^*|_{g_0}^2 d\mu_{g_0}}$$

and using inequality (3.12) and once more (3.3), we eventually get

$$(3.14) \quad \frac{m_0}{2M_0 L^2} (\lambda^*)^4 a(\lambda^*) < \varepsilon^* < \frac{6M_0}{m_0}.$$

In particular, ε^* is positive. (Recall that $M_0 := \max_{B_1(0)} e^{2v_0}$).

For $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$ consider the function $u^* - \varepsilon \varphi^* \in H^1(M; g_0)$. Recall that by (3.7), we trivially have

$$u^* - \varepsilon \varphi^* \in \mathcal{E}_{I(u^* - \varepsilon \varphi^*)}.$$

Lemma 3.9. *For $\varepsilon \in (0, \varepsilon^*)$ we have*

$$(3.15) \quad \beta_{I(u^* - \varepsilon \varphi^*)} < \beta_{\lambda^*}.$$

Proof. By expanding the Dirichlet energy, for $\varepsilon \in (0, \varepsilon^*)$ we obtain

$$\begin{aligned} \beta_{I(u^* - \varepsilon \varphi^*)} \leq E(u^* - \varepsilon \varphi^*) &= E(u^*) - 2\varepsilon \int_M (\nabla u^*, \nabla \varphi^*)_{g_0} d\mu_{g_0} + \varepsilon^2 \int_M |\nabla \varphi^*|_{g_0}^2 d\mu_{g_0} \\ &= E(u^*) - \varepsilon(\varepsilon^* - \varepsilon) \int_M |\nabla \varphi^*|_{g_0}^2 d\mu_{g_0} \\ &< E(u^*) = \beta_{\lambda^*}, \end{aligned}$$

as claimed. \square

The next step is to understand whether the value $I(u^* - \varepsilon \varphi^*)$ is greater or smaller than $\lambda^* = I(u^*)$. In order to do that, we introduce the function $h : [-\varepsilon^*, \varepsilon^*] \rightarrow (0, -\min_M f_0)$ given by:

$$(3.16) \quad h(\varepsilon) := I(u^* - \varepsilon \varphi^*).$$

By definition of I , we have $h \in C^1([-\varepsilon^*, \varepsilon^*])$; moreover, there holds:

Lemma 3.10. *We have that*

$$h' > 0 \text{ on } [0, \varepsilon^*].$$

As a consequence, h is smoothly invertible on $[0, \varepsilon^]$.*

Postponing the proof of the lemma, we continue with the proof of Proposition 3.7.

In view of Lemma 3.10, we have $h(\varepsilon^*) > \lambda^*$. Furthermore, for any $\lambda \in (\lambda^*, h(\varepsilon^*))$ there exists a unique $\varepsilon \in (0, \varepsilon^*)$ such that $h(\varepsilon) = I(u^* - \varepsilon \varphi^*) = \lambda$. From Lemma 3.9, then we get $\beta_\lambda < \beta_{\lambda^*}$.

Therefore, setting $\ell(\lambda^*) := h(\varepsilon^*)$, we obtain the first part of Proposition 3.7.

It remains to show the estimate on the length of this interval $(\lambda^*, \ell(\lambda^*))$ and the relations between it and $(\lambda, \ell(\lambda))$, for $\lambda > \lambda^*$. This will be done in Lemma 3.11.

Proof of Lemma 3.10. Recall that $h(0) = \lambda^*$. Compute the first derivative of h , using (3.8):

$$h'(\varepsilon) = DI(u^* - \varepsilon\varphi^*)[-\varphi^*] = 2 \frac{\int_M f_{I(u^* - \varepsilon\varphi^*)} \varphi^* e^{2u^* - 2\varepsilon\varphi^*} d\mu_{g_0}}{\int_M e^{2u^* - 2\varepsilon\varphi^*} d\mu_{g_0}}.$$

Thus

$$h'(0) = 2 \frac{\int_M f_{\lambda^*} \varphi^* e^{2u^*} d\mu_{g_0}}{\int_M e^{2u^*} d\mu_{g_0}} > 0$$

in view of (3.12). By continuity of h' , there exists $\varepsilon \in (0, \varepsilon^*]$ such that $h' > 0$ on $[0, \varepsilon)$ and such that ε is maximal with this property. We claim that $\varepsilon = \varepsilon^*$. Suppose by contradiction that $\varepsilon < \varepsilon^*$. Note that $h(\varepsilon) = I(u^* - \varepsilon\varphi^*) > \lambda^* = h(0)$, since $h' > 0$ on $[0, \varepsilon)$. Moreover,

$$\begin{aligned} \int_M f_{h(\varepsilon)} \varphi^* e^{2u^* - 2\varepsilon\varphi^*} d\mu_{g_0} &= \int_{B_{\frac{\sqrt{\lambda^*}}{L}}(0)} f_{h(\varepsilon)} \varphi^* e^{2u^* - 2\varepsilon\varphi^*} e^{2v_0} dx \\ &\geq \frac{h(\varepsilon)}{2} \int_{B_{\frac{\sqrt{\lambda^*}}{L}}(0)} \varphi^* e^{(2u^* - 2\varepsilon\varphi^*)} e^{2v_0} dx \end{aligned}$$

where in the last inequality we used the fact that

$$f_{h(\varepsilon)} \geq \frac{h(\varepsilon)}{2} \text{ on } B_{\frac{\sqrt{h(\varepsilon)}}{L}}(0) \supset B_{\frac{\sqrt{\lambda^*}}{L}}(0)$$

(recall Lemma 3.2). Therefore, we obtain

$$\begin{aligned} \int_M f_{h(\varepsilon)} \varphi^* e^{2u^* - 2\varepsilon\varphi^*} d\mu_{g_0} &\geq \frac{h(\varepsilon)}{2} \log(1/\lambda^*) \int_{B_{\frac{(\lambda^*)^{3/2}}{L}}(0)} e^{(2u^* - 2\varepsilon\varphi^*)} e^{2v_0} dx \\ &> \frac{\lambda^*}{2} \log(1/\lambda^*) \int_{B_{\frac{(\lambda^*)^{3/2}}{L}}(0)} e^{(2u^* - 2\varepsilon^*\varphi^*)} e^{2v_0} dx \\ &= \frac{(\lambda^*)^{1+2\varepsilon^*}}{2} \log(1/\lambda^*) \int_{B_{\frac{(\lambda^*)^{3/2}}{L}}(0)} e^{2u^*} e^{2v_0} dx \\ &\geq \frac{m_0\pi}{2L^2} \log(1/\lambda^*) (\lambda^*)^{4+2\varepsilon^*} a(\lambda^*) > 0 \end{aligned}$$

where in the last line we used (3.11). Thus, we have, since $\varepsilon > 0$ and $\varphi^* \geq 0$,

$$\begin{aligned} h'(\varepsilon) &> \frac{m_0\pi}{L^2} \log(1/\lambda^*) (\lambda^*)^{4+2\varepsilon^*} \frac{a(\lambda^*)}{\int_M e^{2u^* - 2\varepsilon\varphi^*} d\mu_{g_0}} \\ (3.17) \quad &> \frac{m_0\pi}{L^2} \log(1/\lambda^*) (\lambda^*)^{4+2\varepsilon^*} \frac{a(\lambda^*)}{\int_M e^{2u^*} d\mu_{g_0}} > 0, \end{aligned}$$

contradicting the maximality of ε . Furthermore, reasoning as we have just done, we see that the bound (3.17) holds uniformly on $(0, \varepsilon^*)$. We deduce $h'(\varepsilon^*) > 0$ and the Lemma is proved. \square

Lemma 3.11. *Let λ_0 be defined as in the proof of Proposition 3.7. Fix $0 < \lambda^* < \lambda < \lambda_0$ and consider $\ell(\lambda)$ given by the first part of Proposition 3.7. Then*

$$\ell(\lambda) - \lambda \geq \tau = \tau(\lambda^*) > 0$$

where τ is a constant not depending on λ .

Proof of Lemma 3.11. Let's begin with estimating $\ell(\lambda^*) - \lambda^*$. We restart from (3.17), which holds for $\varepsilon \in (0, \varepsilon^*)$. By equation (3.14) and by the fact that $\lambda^* < 1$, we get $(\lambda^*)^{4+2\varepsilon^*} > (\lambda^*)^{4+\frac{12M_0}{m_0}}$ and $\log(1/\lambda^*) > \log(1/\lambda_0)$. Recalling the definition of the auxiliary function A (equation (3.9)), we can bound

$$A(\lambda^*) \geq \int_M e^{2u^*} d\mu_{g_0}$$

and obtain

$$h'(\varepsilon) > \frac{m_0\pi}{L^2} \log(1/\lambda_0) (\lambda^*)^{4+\frac{12M_0}{m_0}} \frac{a(\lambda^*)}{A(\lambda^*)}.$$

Recalling once more (3.14), with the constant $k_0 := \frac{m_0^2\pi}{2M_0L^4} \log(1/\lambda_0) > 0$ we may finally estimate

$$\begin{aligned} \ell(\lambda^*) - \lambda^* = h(\varepsilon^*) - \lambda^* &= \int_0^{\varepsilon^*} h'(\varepsilon) d\varepsilon \\ &> \varepsilon^* \frac{m_0\pi}{L^2} \log(1/\lambda_0) (\lambda^*)^{4+\frac{12M_0}{m_0}} \frac{a(\lambda^*)}{A(\lambda^*)} \\ &> k_0 (\lambda^*)^{8+\frac{12M_0}{m_0}} \frac{(a(\lambda^*))^2}{A(\lambda^*)}, \end{aligned}$$

and the function $(\lambda^*)^{8+\frac{12M_0}{m_0}} \frac{(a(\lambda^*))^2}{A(\lambda^*)}$ is not decreasing in λ^* .

Hence, taking $\lambda \in (\lambda^*, \lambda_0)$, we deduce

$$\ell(\lambda) - \lambda > k_0 (\lambda)^{8+\frac{12M_0}{m_0}} \frac{(a(\lambda))^2}{A(\lambda)} \geq k_0 (\lambda^*)^{8+\frac{12M_0}{m_0}} \frac{(a(\lambda^*))^2}{A(\lambda^*)} := \tau > 0.$$

The Lemma is proved. \square

This concludes the proof of Proposition 3.7. \square

3.2. A bound for the total curvature.

With the help of Corollary 3.8, Proposition 3.3 and following [20], it is now quite straightforward to show the following estimate for the derivative of β_λ :

Lemma 3.12. *There exists a sequence $(\lambda_n)_n \subset (0, \lambda_0)$ of points of differentiability for β_λ , such that $\lambda_n \downarrow 0$ as $n \rightarrow \infty$ and*

$$|\beta'_{\lambda_n}| \leq C_0/\lambda_n$$

where C_0 is a positive constant.

Proof. By Proposition 3.3, we have $\beta_\lambda \leq C \log(1/\lambda)$, for any $\lambda < \lambda_0$. Set $C_0 := C + 1$ and assume that exists $\tilde{\lambda} < \lambda_0$ such that for any $\lambda < \tilde{\lambda}$, λ point of differentiability of β_λ , there holds:

$$|\beta'_\lambda| > C_0/\lambda.$$

Then we obtain, by Lebesgue's Theorem, that

$$\beta_\lambda - \beta_{\tilde{\lambda}} \geq \int_{\tilde{\lambda}}^{\lambda} |\beta'_s| ds$$

and hence

$$C \log(1/\lambda) \geq \beta_\lambda > \beta_{\tilde{\lambda}} + C_0 \log(\tilde{\lambda}/\lambda).$$

Thus, we get

$$\beta_{\tilde{\lambda}} + C_0 \log(\tilde{\lambda}) - \log(\lambda) \leq 0,$$

which, for λ small enough, is clearly impossible. The Lemma is proved. \square

We can now prove the analogue of equation (5.1) in [5]:

Proposition 3.13. *Let $(\lambda_n)_n$ be a sequence like the one given by Lemma 3.12. and set $u_n := u_{\lambda_n}$. Then*

$$(3.18) \quad \limsup_n \left(\lambda_n \int_M e^{2u_n} d\mu_{g_0} \right) < \infty$$

Proof. Fix $n \in \mathbb{N}$ and set for convenience $\lambda^* := \lambda_n \in (0, \lambda_0)$ and $u^* := u_n$.

Consider the function h defined by equation (3.16), where ε^* and φ^* are defined as in the proof of Proposition 3.7. For $\lambda_k \downarrow \lambda^*$, $\lambda_k < h(\varepsilon^*)$, set $\varepsilon_k := h^{-1}(\lambda_k)$. By Lemma 3.10, $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Finally, by Lemma 3.12, we may assume that for all k

$$-\frac{\beta_{\lambda^*} - \beta_{\lambda_k}}{\lambda^* - \lambda_k} \leq 2C_0/\lambda^* := C/\lambda^*$$

where C_0 is the constant of Lemma 3.12.

Observe that $\beta_{\lambda^*} - \beta_{\lambda_k} \geq E(u^*) - E(u^* - \varepsilon_k \varphi^*)$, since $u^* - \varepsilon_k \varphi^* \in \mathcal{E}_{I(u^* - \varepsilon_k \varphi^*)} = \mathcal{E}_{\lambda_k}$. Now:

$$E(u^*) - E(u^* - \varepsilon_k \varphi^*) = \int_M (-\varepsilon_k^2 |\nabla \varphi^*|_{g_0}^2 + 2\varepsilon_k (\nabla u^*, \nabla \varphi^*)_{g_0}) d\mu_{g_0}.$$

Hence,

$$\frac{C}{\lambda^*} \geq \frac{1}{\lambda_k - \lambda^*} \int_M (-\varepsilon_k^2 |\nabla \varphi^*|_{g_0}^2 + 2\varepsilon_k (\nabla u^*, \nabla \varphi^*)_{g_0}) d\mu_{g_0}.$$

Recalling that $\varepsilon_k = h^{-1}(\lambda_k)$, $h^{-1}(\lambda^*) = 0$ and using (3.3), we have

$$\begin{aligned} \frac{1}{\lambda_k - \lambda^*} \int_M \varepsilon_k^2 |\nabla \varphi^*|_{g_0}^2 &\leq 2\pi M_0 \log(1/\lambda^*) \frac{h^{-1}(\lambda_k) - h^{-1}(\lambda^*)}{\lambda_k - \lambda^*} \varepsilon_k \\ &\rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$, since h^{-1} is differentiable at λ^* and ε_k goes to zero. Therefore, we may write, with an error term $o(1)$ as $k \rightarrow \infty$, that

$$\frac{C}{\lambda^*} \geq \frac{2\varepsilon_k}{\lambda_k - \lambda^*} \int_M (\nabla u^*, \nabla \varphi^*)_{g_0} d\mu_{g_0} + o(1).$$

Thus, when $k \rightarrow \infty$, we obtain

$$\begin{aligned} \frac{C}{\lambda^*} &\geq 2(h^{-1})'(\lambda^*) \int_M (\nabla u^*, \nabla \varphi^*)_{g_0} d\mu_{g_0} \\ &= \frac{2}{h'(0)} \int_M f_{\lambda^*} \varphi^* e^{2u^*} d\mu_{g_0} \\ &= \int_M e^{2u^*} d\mu_0 \end{aligned}$$

where in the last line we have used the explicit expression of $h'(0)$. Going back to the original notation, we have for any $n \in \mathbb{N}$

$$\int_M e^{2u_n} d\mu_{g_0} \leq C/\lambda_n,$$

which is nothing but equation (3.18). The Proposition is proved. \square

As a consequence of Proposition 3.13 and the Gauss-Bonnet identity $0 = \int_M f_{\lambda_n} e^{2u_n} d\mu_{g_0}$, we deduce the uniform bound

$$\sup_{n \in \mathbb{N}} \int_M (|f_0| + \lambda_n) e^{2u_n} d\mu_{g_0} < \infty$$

for the total curvature of $g_n = e^{2u_n} g_0$.

3.3. Blow-up analysis.

In this subsection we complete the Proof of Theorem 1.1. For the rest of this part, let $(\lambda_n)_n$ be a sequence like the one given by Lemma 3.12 and set $u_n := u_{\lambda_n}$. We follow closely Section 5 of [5].

As shown by Ding-Liu [11], we obtain for any open domain $\Omega \subset \subset M^- := \{p \in M : f_0(p) < 0\}$, $\int_{\Omega} (|\nabla u_n^+|_{g_0}^2 + |u_n^+|^2) d\mu_{g_0} \leq C(\Omega)$, where $t^+ = \max\{t, 0\}$, $t \in \mathbb{R}$, and hence, as proved in [5], that

$$(3.19) \quad u_n \leq C'(\Omega).$$

Thus, if a sequence $(u_n)_n$ blows up near a point $p_0 \in M$ in the sense that for every $r > 0$ there holds $\sup_{B_r(p_0)} u_n \rightarrow +\infty$ (and we know that it is always the case in view of Remark 3.6), necessarily $f_0(p_0) = 0$. Moreover, there exists a sequence of points $p_n \rightarrow p_0$ such that for some $r > 0$, $u_n(p_n) = \sup_{B_r(p_0)} u_n$.

Let p_0 be such a blow-up point for a sequence of solutions u_n . We introduce local isothermal coordinates x on $B_r(p_0)$ around $p_0 = 0$. We can write $g_0 = e^{2v_0} g_{\mathbb{R}^2}$ for some smooth function v_0 . Setting $v_n := u_n + v_0$, we get

$$-\Delta v_n = (f_0(x) + \lambda_n) e^{2v_n} \quad \text{on } B_R(0)$$

for some $R > 0$ and there is a sequence $x_n \rightarrow 0$ so that

$$v_n(x_n) = \sup_{|x| \leq R} v_n(x) \rightarrow +\infty$$

as $n \rightarrow +\infty$. Moreover, $\Delta v_n(x_n) \leq 0$ and thus $f_0(x_n) + \lambda_n \geq 0$, which leads to

$$|x_n|^2 \leq C\lambda_n$$

for some constant $C > 0$.

We observe that in the present case we do not have available a uniform global lower bound for the sequence of solutions u_n (and hence for v_n) of the kind present in [5]. But we can still show that the analogue of Lemma 5.2 [5] holds true. Indeed, a careful inspection shows that a uniform lower bound is not needed in the proof of Lemma 5.2 [5].

Lemma 3.14. *For every $r > 0$, that holds*

$$\limsup_n \int_{B_r(0)} (f_0 + \lambda_n)^+ e^{2v_n} dx \geq 2\pi.$$

In order to prove Theorem 1.1, as regards part ii), we would like to imitate the proof of Theorem 1.4 [5]. To do that and to show the convergence results therein, the last ingredient we need is at least a local lower bound for our sequence of solutions u_n .

The next Lemma shows that either the sequence degenerates or that we have a local lower bound. After this Lemma, we will obtain part i) of Theorem 1.1. To prove part ii), it will be sufficient to repeat the same reasoning as after Lemma 5.2. in [5].

Lemma 3.15. *Let $(\lambda_n)_n$ and $(u_n)_n$ be defined as above and set*

$$M_\infty := M \setminus \left\{ p_\infty^{(1)}, \dots, p_\infty^{(I)} \right\}.$$

where $p_\infty^{(1)}, \dots, p_\infty^{(I)}$ are blow-up points. Then, up to subsequences, either
i) $u_n \rightarrow -\infty$ locally uniformly on compact domains of M_∞ , or
ii) for any compact domain $\Omega \subset\subset M_\infty$, there exists a constant $C = C(\Omega) \in \mathbb{R}$ such that

$$u_n|_\Omega > C(\Omega)$$

uniformly in n .

Proof. We fix two open domains $\Omega \subset\subset \tilde{\Omega} \subset\subset M_\infty$. From (3.19), for any n we get that $u_n|_{\tilde{\Omega}} \leq C(\tilde{\Omega})$. We pick an arbitrary point $p \in \tilde{\Omega}$ and $r_p > 0$ so that $B_{r_p}(p) \subset \tilde{\Omega}$. If needed, we choose a smaller radius and we consider a conformal chart $\Psi : B_{r_p}(p) \rightarrow B_1(0) \subset \mathbb{R}^2$ with coordinates x so that locally we have $g_0 = e^{2v_0} g_{\mathbb{R}^2}$ with $v_0 \in C^\infty(\overline{B_1(0)})$. Setting $v_n := u_n + v_0$, we obtain

$$-\Delta v_n = (f_0(x) + \lambda_n) e^{2v_n} \quad \text{on } B_1(0).$$

Split $v_n = v_n^{(0)} + v_n^{(1)}$, where $v_n^{(1)} \in H_0^1(B_1(0))$ solves the boundary value problem

$$\begin{cases} -\Delta v_n^{(1)} = (f_0(x) + \lambda_n) e^{2v_n} & \text{in } B_1(0), \\ v_n^{(1)} = 0 & \text{on } \partial B_1(0). \end{cases}$$

and $v_n^{(0)}$ is harmonic. Hence it follows, uniformly in n ,

$$\|\Delta v_n^{(1)}\|_{L^p(B_1(0))} \leq \|\Delta v_n^{(1)}\|_{L^\infty(B_1(0))} \leq C$$

for any $p \geq 1$. Fixing $p > 1$, from elliptic regularity theory we obtain that $(v_n^{(1)})_n$ is bounded in $W^{2,p}(B_1(0)) \hookrightarrow C^0(\overline{B_1(0)})$. From the local upper bound on $\tilde{\Omega}$ for the sequence $(u_n)_n$ (and hence for $(v_n)_n$), we infer that for any $x \in \overline{B_1(0)}$,

$$v_n^{(0)}(x) \leq \|v_n^{(1)}\|_{L^\infty(B_1(0))} + C(\tilde{\Omega}) \leq C$$

uniformly in n . Therefore, Harnack's inequality implies that

$$\sup_{B_{1/2}(0)} v_n^{(0)} \leq C_1 \inf_{B_{1/2}(0)} v_n^{(0)} + C_2$$

for suitable constants $C_1 > 0$ and $C_2 \in \mathbb{R}$ depending on $B_{1/2}(0)$ but not on n .

We see that we have two mutually disjoint cases (up to subsequences):

- i. $\inf_{B_{1/2}(0)} v_n^{(0)} \rightarrow -\infty$, as $n \rightarrow +\infty$
- ii. $\inf_{B_{1/2}(0)} v_n^{(0)} \geq -C$, uniformly in n .

In the first case, it follows, recalling that $(v_n^{(1)})_n$ is bounded in $L^\infty(B_1(0))$, that

$$v_n \rightarrow -\infty$$

uniformly in $\overline{B_{1/2}(0)}$.

In the second case, we deduce $C < v_n|_{\overline{B_{1/2}(0)}}$ uniformly in n .

Since $\overline{\Omega}$ is connected, we conclude that either on $\overline{\Omega}$ the sequence of solutions u_n goes uniformly to $-\infty$ or that there exists $C = C(\Omega)$ such that $u_n|_\Omega > C$ for any n . The Lemma is proved. \square

4. PROOF OF THEOREM 1.2

In this section, we will analyze the asymptotic behaviour of the set of solutions to the prescribed Gaussian curvature equation, when the parameter $\lambda \uparrow -\overline{f_0} = \lambda_{max}$. The main content of this section is the proof of Theorem 1.2.

Proposition 4.1. *Let β_λ be defined by equation (2.3). Then $\beta_\lambda \rightarrow 0$ as $\lambda \uparrow \lambda_{max}$.*

In preparation for the proof of the Proposition, consider the Hilbert space $H^1(M; g_0) \times \mathbb{R}$ endowed with the natural scalar product and consider the set

$$(4.1) \quad \mathcal{C} := \left\{ (u, \lambda) \in H^1(M; g_0) \times \mathbb{R} : \int_M u \, d\mu_{g_0} = 0 = \int_M f_\lambda e^{2u} \, d\mu_{g_0} \right\}.$$

We claim that \mathcal{C} is a C^∞ -Banach manifold. Indeed, we define $G : H^1(M; g_0) \times \mathbb{R} \rightarrow \mathbb{R}^2$ as:

$$G(u, \lambda) := \left(\int_M u \, d\mu_{g_0} ; \int_M f_\lambda e^{2u} \, d\mu_{g_0} \right).$$

Then

$$G^{-1}((0, 0)) = \mathcal{C}$$

and $G \in C^\infty$ with first Frechet derivative

$$DG(u, \lambda) [v, t] = \left(\int_M v \, d\mu_{g_0} ; 2 \int_M f_\lambda v e^{2u} \, d\mu_{g_0} + t \int_M e^{2u} \, d\mu_{g_0} \right)$$

for any $(v, t) \in H^1(M; g_0) \times \mathbb{R}$.

For any $(u, \lambda) \in \mathcal{C}$, letting $DG(u, \lambda)$ act on $(1, 0)$ and $(0, 1)$, we obtain respectively the vectors $(1, 0)$ and $(0, \int_M e^{2u} \, d\mu_{g_0})$, which are clearly a basis for \mathbb{R}^2 . Moreover, the kernel of $DG(u, \lambda)$ splits $H^1(M; g_0) \times \mathbb{R}$. Thus, \mathcal{C} is a smooth manifold of codimension equal to 2.

Define

$$\tilde{\mathcal{C}}_\lambda := \mathcal{C} \cap \{(w, \mu) \in H^1(M; g_0) \times \mathbb{R} : \mu = \lambda\}$$

that is, the slice of \mathcal{C} determined by the hyperplane in $H^1(M; g_0) \times \mathbb{R}$ of equation $\mu = \lambda$. We observe that this set is not empty for $\lambda \in (0, -\min_M f_0)$.

Lemma 4.2. *There exist a function $s : \mathcal{C}_{\lambda_{max}} \rightarrow \mathbb{R}$ and a map $\Theta : \mathcal{C}_{\lambda_{max}} \times (0, -\min_M f_0) \rightarrow H^1(M; g_0)$ such that for any $(u, \lambda) \in \mathcal{C}_{\lambda_{max}} \times (0, -\min_M f_0)$ we have*

$$u + s(u)(\lambda - \lambda_{max})(f_0 - \overline{f_0}) + \Theta(u, \lambda) \in \mathcal{C}_\lambda$$

and with the property that for any fixed $u \in \mathcal{C}_{\lambda_{max}}$

$$\|\Theta(u, \lambda)\|_{H^1(M; g_0)} = o(\lambda - \lambda_{max})$$

as $\lambda \rightarrow \lambda_{max}$.

Proof. We take $u \in \mathcal{C}_{\lambda_{max}}$, $\lambda \in (0, -\min_M f_0)$ and consider the vector $(s(f_0 - \overline{f_0}), 1) \in H^1(M; g_0) \times \mathbb{R}$ where $s \in \mathbb{R}$. We want to find a suitable $s = s(u)$ such that the vector $(s(f_0 - \overline{f_0}), 1)$ belongs to the tangent space $T_{(u, \lambda_{max})}\mathcal{C}$.

That amounts to impose

$$DG(u, \lambda_{max}) [s(f_0 - \overline{f_0}), 1] = (0, 0)$$

that is,

$$\begin{pmatrix} s \int_M (f_0 - \overline{f_0}) \, d\mu_{g_0} \\ 2s \int_M (f_0 - \overline{f_0})^2 e^{2u} \, d\mu_{g_0} + \int_M e^{2u} \, d\mu_{g_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $\int_M (f_0 - \overline{f_0}) d\mu_{g_0} = 0$, we get from the second equation that

$$s(u) = -\frac{\int_M e^{2u} d\mu_{g_0}}{2 \int_M (f_0 - \overline{f_0})^2 e^{2u} d\mu_{g_0}} < 0.$$

In view of the differentiable structure of \mathcal{C} , there exists $\Theta : \mathcal{C}_{\lambda_{max}} \times (0, -\min_M f_0) \rightarrow H^1(M; g_0)$ such that

$$(u + s(u)(\lambda - \lambda_{max})(f_0 - \overline{f_0}) + \Theta(u, \lambda); \lambda) \in \tilde{\mathcal{C}}_\lambda$$

and $\|\Theta(u, \lambda)\|_{H^1(M; g_0)} = o(\lambda - \lambda_{max})$ as $\lambda \rightarrow \lambda_{max}$. The result follows. \square

Proof of Proposition 4.1. We choose $u \equiv 0 \in \mathcal{C}_{\lambda_{max}}$, $\lambda \in (0, -\min_M f_0)$ and compute s and Θ accordingly. Thus, $v_\lambda := s(0)(\lambda - \lambda_{max})(f_0 - \overline{f_0}) + \Theta(0, \lambda) \in \mathcal{C}_\lambda$; we evaluate its $H^1(M; g_0)$ norm

$$\begin{aligned} & \|s(0)(\lambda - \lambda_{max})(f_0 - \overline{f_0}) + \Theta(0, \lambda)\|_{H^1(M; g_0)} \leq \\ & \leq |s(0)| |\lambda - \lambda_{max}| \|f_0 - \overline{f_0}\|_{H^1(M; g_0)} + o(\lambda - \lambda_{max}) \end{aligned}$$

and see that it goes to zero as $\lambda \rightarrow \lambda_{max}$.

Since for $\lambda < \lambda_{max}$ we have by definition $\beta_\lambda \leq E(v_\lambda)$, it follows $\beta_\lambda \rightarrow 0$ as $\lambda \uparrow \lambda_{max}$. \square

Proof of Theorem 1.2 (completed). Let $w_\lambda \in \mathcal{C}_\lambda$ be a minimizer for $\lambda \in \Lambda$, as the one given in Section 2: then, since $\overline{w}_\lambda = 0$ and $\|\nabla w_\lambda\|_{L^2(M)}^2 = \beta_\lambda \rightarrow 0$ when $\lambda \uparrow \lambda_{max}$, it follows by Poincaré-Wirtinger's inequality that $w_\lambda \rightarrow 0$ in $H^1(M; g_0)$.

Applying Moser-Trudinger's inequality, we also have $e^{2w_\lambda} \rightarrow 1$ in $L^p(M)$ for any $p \in [1, \infty)$. Therefore, by Hölder's inequality, we obtain that for any $v \in H^1(M; g_0)$

$$\int_M f_\lambda v e^{2w_\lambda} d\mu_{g_0} \rightarrow \int_M (f_0 - \overline{f_0}) v d\mu_{g_0}$$

when $\lambda \uparrow \lambda_{max}$. We recall that, for any $\lambda \in \Lambda$, w_λ solves

$$\int_M (\nabla w_\lambda, \nabla v)_{g_0} d\mu_{g_0} = \mu(\lambda) \int_M f_\lambda v e^{2w_\lambda} d\mu_{g_0}, \quad v \in H^1(M; g_0)$$

where $\mu(\lambda) > 0$ is a Lagrange multiplier. Choosing $v = f_0 - \overline{f_0}$, we obtain for $\lambda \uparrow \lambda_{max}$

$$0 = \lim_{\lambda \uparrow \lambda_{max}} \mu(\lambda) \int_M (f_0 - \overline{f_0})^2 d\mu_{g_0}$$

and therefore $\lim_{\lambda \uparrow \lambda_{max}} \mu(\lambda) = 0$.

Thus, using L^p -estimates, we obtain

$$\|w_\lambda\|_{H^2(M)} \leq c (\|\Delta w_\lambda\|_{L^2(M)} + \|w_\lambda\|_{H^1(M)}).$$

Since

$$\|\mu(\lambda) f_\lambda e^{2w_\lambda}\|_{L^2(M)} \leq \mu(\lambda) \|f_\lambda\|_\infty \left[\int_M e^{4w_\lambda} d\mu_{g_0} \right]^{1/2}$$

and $e^{4w_\lambda} \rightarrow 1$ in L^1 as $\lambda \uparrow \lambda_{max}$, it follows that $\|\Delta w_\lambda\|_{L^2(M)} \rightarrow 0$ and hence w_λ converges to zero in $H^2(M, g_0)$. By Sobolev's embedding results, we also have for any $\alpha \in [0, 1)$

$$w_\lambda \rightarrow 0 \text{ in } C^{0, \alpha}(M)$$

when $\lambda \uparrow \lambda_{max}$.

Thus, using the bootstrap method and Schauder's estimates, we obtain $C^{2, \alpha}$ convergence as well.

Finally, we obtain that

$$u_\lambda := w_\lambda + 1/2 \log \mu(\lambda),$$

solution to equation (1.1), goes uniformly to $-\infty$ on M when $\lambda \uparrow \lambda_{max}$ and therefore it can not admit any convergent subsequence.

This concludes the proof of Theorem 1.2. \square

Remark 4.3. Because of the conformal invariance of the Dirichlet energy and from convergence $\|\nabla u_\lambda\|_{L^2(M)}^2 \rightarrow 0$ as $\lambda \uparrow -\bar{f}_0 = \lambda_{max}$, it follows that no “fine structure” can appear in the “limit” geometry of the surfaces $(M, e^{2u_\lambda} g_0)$, independently of how we blow up the scale.

REFERENCES

- [1] Aubin, Thierry: *Some Nonlinear Problems in Riemannian Geometry*. Springer-Verlag Berlin Heidelberg (1998)
- [2] Aubin, Thierry: *Sur le problème de la courbure scalaire prescrite*. (French. English, French summary) [On the problem of prescribed scalar curvature] Bull. Sci. Math. 118 (1994), no. 5, 465-474.
- [3] Berger, Melvyn S.: *Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds*. J. Differential Geometry 5 (1971), 325-332.
- [4] Bismuth, Sophie: *Prescribed scalar curvature on a C^∞ compact Riemannian manifold of dimension two*. Bull. Sci. Math. 124 (2000), no. 3, 239-248.
- [5] Borer, Franziska; Galimberti, Luca; Struwe, Michael: “Large” conformal metrics of prescribed Gauss curvature on surfaces of higher genus. Comm. Math. Helv. (to appear)
- [6] Brezis, Haim; Merle, Frank: *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*. Comm. Partial Differential Equations 16 (1991), no. 8-9, 1223-1253.
- [7] Chang, Sun-Yung Alice: *Non-linear elliptic equations in conformal geometry*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2004.
- [8] Chang, Kung Ching; Liu, Jia Quan: *A Morse-theoretic approach to the prescribing Gaussian curvature problem*. Variational methods in nonlinear analysis (Erice, 1992), 55-62, Gordon and Breach, Basel, 1995.
- [9] Chen, Wen Xiong; Li, Congming: *Classification of solutions of some nonlinear elliptic equations*. Duke Math. J. 63 (1991), 615-622.
- [10] Cheng, Kuo-Shung; Lin, Chang-Shou: *On the asymptotic behavior of solutions of the conformal Gaussian curvature equations in \mathbb{R}^2* . Math. Ann. 308 (1997), 119-139.
- [11] Ding, Wei Yue; Liu, Jiaquan: *A note on the prescribing Gaussian curvature on surfaces*, Trans. Amer. Math. Soc. 347 (1995), 1059-1066.
- [12] Huber, Alfred: *On subharmonic functions and differential geometry in the large*. Comment. Math. Helv. 32 (1957), 13-72.
- [13] Kazdan, Jerry L.; Warner, F. W.: *Curvature functions for compact 2-manifolds*. Ann. of Math. (2) 99 (1974), 14-47.
- [14] Kazdan, Jerry L.; Warner, F. W.: *Scalar curvature and conformal deformation of Riemannian structure*. J. Differential Geometry 10 (1975), 113-134.
- [15] Li, Yan Yan; Shafrir, Itai: *Blow-up analysis for solutions of $-\Delta u = V(x)e^u$ in dimension two*. Indiana Univ. Math. J. 43 (1994), no. 4, 1255-1270.
- [16] Martinazzi, Luca: *Concentration-compactness phenomena in the higher order Liouville’s equation*. J. Funct. Anal. 256 (2009), no. 11, 3743-3771.
- [17] Moser, J.: *On a nonlinear problem in differential geometry*. Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), pp. 273-280. Academic Press, New York, 1973.
- [18] Struwe, Michael: *Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces*. Ann. Inst. H. Poincaré Anal. Non Linéaire 5 (1988), no. 5, 425-464.
- [19] Struwe, Michael: *The existence of surfaces of constant mean curvature with free boundaries*. Acta Math. 160 (1988), no. 1-2, 19-64.
- [20] Struwe, Michael: *Une estimation asymptotique pour le modèle de Ginzburg-Landau*. [An asymptotic estimate for the Ginzburg-Landau model] C. R. Acad. Sci. Paris Sr. I Math. 317 (1993), no. 7, 677-680.

- [21] Struwe, Michael; Tarantello, Gabriella: *On multivortex solutions in Chern-Simons gauge theory*. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), no. 1, 109-121.
- [22] Zeidler, Eberhard: *Nonlinear Functional Analysis and its Applications III*. Springer-Verlag New York (1985)

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